MTH 301 Final Solutions

1. (a) Show that the cycles $(1\,2\,3)$ and $(1\,3\,2)$ are not conjugate in A_4.

   (b) Verify that the Class Equation for A_4 has the form

$$12 = 1 + 3 + 4 + 4.$$

Solution. (a) The elements of A_4 may be enumerated as

$$\{(1), (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$$

By a direct computation, one can see that the conjugacy class of the cycle (123) is

$$\{(123), (243), (142), (134)\}.$$

Hence, the cycles (123) and (132) are not conjugate.

(b) Noting that conjugate elements in S_n have the same partition type, a simple computation reveals that A_4 has 3 other conjugacy classes, which are:

$$\{(1)\}, \{(12)(34), (13)(24), (14)(23)\} \text{ and } \{(132), (124), (143), (234)\}.$$

As the conjugacy class representatives (123) and (132) do not commute with the representative $(12)(34)$ we see that $Z(G) = \{(1)\}$. Hence, the class equation for A_4 takes the form

$$12 = 1 + 3 + 4 + 4.$$

2. For distinct primes p and q, show that a group of order p^2q is not simple.

Solution. Let G be a group of order p^2q. By the First Sylow Theorem, G has subgroups of orders p, p^2 and q. Suppose that $p > q$. Then a subgroup H of order p^2 has index q in G, and hence $H \trianglelefteq G$, which proves that G is not simple.

Suppose that $p < q$. The Third Sylow Theorem implies that

$$n_p \equiv 1 \pmod{p} \text{ and } n_p \mid q \tag{1}$$
$$n_q \equiv 1 \pmod{q} \text{ and } n_q \mid p^2 \tag{2}$$

From (2) above, we see that for some integer k, $qk+1 \mid p^2$, which would imply that $q \mid p^2 - 1$. Since q is prime and $p < q$, we have $q \mid p+1$ or $q = p+1$. Hence, $p = 2$ and $q = 3$ and $|G| = 12$. The result now follows from the fact that a group of order 12 is non-simple.

3. Let G be a group of order 231. Show that G has a subgroup subgroup H of order 11 such that $H \leq Z(G)$.

   Solution. The First Sylow Theorem would imply that G has a subgroup H of order 11, while the Third Sylow Theorem would yield

   $$n_{11} \equiv 1 \pmod{11} \text{ and } n_{11} \mid 21.$$

   Hence $n_{11} = 1$, or in other words, H is the unique Sylow-11 subgroup of G, and so $H \trianglelefteq G$.

   Now consider the action $G \curvearrowright^c H$. This induces a permutation representation

   $$\psi_c : G \to \mathrm{Aut}(H)$$

   with $\mathrm{Ker}\,\psi_c = C_G(H)$. Since $H \cong Z_{11}$, we have

   $$|\mathrm{Aut}(H)| = |U_{11}| = 10.$$

   Hence, $|G/C_G(H)| \mid 10$, which would imply that $|G/C_G(H)| = 1$. Therefore, $C_G(H) = G$, that is, $H \leq Z(G)$.

4. Find up to isomorphism:

   (a) The number of distinct abelian groups of order 144.

   (b) Three non-abelian groups of order 30.

   Solution. (a) Up to isomorphism, there are 10 distinct abelian groups of order 144, and they are:

   (1) $\mathbb{Z}_{16} \times \mathbb{Z}_9$

   (2) $\mathbb{Z}_{48} \times \mathbb{Z}_3$

   (3) $\mathbb{Z}_{36} \times \mathbb{Z}_4$

   (4) $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$

   (5) $\mathbb{Z}_2 \times \mathbb{Z}_{72}$

   (6) $\mathbb{Z}_6 \times \mathbb{Z}_{24}$

   (7) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{36}$

   (8) $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{12}$

   (9) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$

   (10) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$

(b) Three non-abelian groups of order 30 are: D_{30}, $S_3 \times \mathbb{Z}_5$, and $D_{10} \times \mathbb{Z}_3$. It is left as an exercise to show that these groups are non-isomorphic. In fact, these are the only non-abelian groups of order 30 up to isomorphism.

5. Without using the Feit-Thompson theorem, show that the following statements are equivalent.

   (i) Every group of odd-order is solvable.

   (ii) The only simple groups of odd order are of prime order.

   Solution. ($\Longrightarrow$) Suppose that every group of odd order is solvable. Let G be a simple group of odd order. By our assumption, G has to be solvable. As G has no proper normal subgroups, G has to be abelian. The Classification Theorem of finite abelian groups would now imply that G has to be of prime order.

   ($\Longleftarrow$) Suppose that the simple groups of odd order are those of prime order. Let H be a group of odd order. If H is simple, then our hypothesis would imply that H is of prime order, and hence solvable. On the contrary, if H is not simple, then let

   $$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_k = H$$

   be a composition series for H. Each factor group H_{i+1}/H_i is a simple group of odd order, and so our hypothesis would imply that it is abelian of prime order. Thus, we have a normal series for H in which each factor group is abelian. Therefore, H is solvable.

6. For positive integers m and n such that $n \mid m$, consider the map

   $$\varphi(m,n) : U_m \to U_n \;:\; [x]_m \xmapsto{\varphi(m,n)} [x]_n.$$

   (a) Show that $\varphi(m,n)$ is a well-defined homomorphism.

   (b) Find $\operatorname{Ker} \varphi(36,12)$ and $\operatorname{Im} \varphi(36,12)$.

   Solution. (a) The map $\varphi(m,n)$ is well-defined, for if $[x]_m = [y]_m$, then

   $$m \mid x - y \implies n \mid x - y \implies [x]_n = [y]_n.$$

   Furthermore, $\varphi(m,n)$ is a homorphism, because

   $$\begin{aligned} \varphi(m,n)([x]_m[y]_m) &= \varphi(m,n)([xy]_m) \\ &= [xy]_n \\ &= [x]_n[y]_n \\ &= \varphi(m,n)([x]_m)\varphi(m,n)([x]_m). \end{aligned}$$

(b) Since for any positive integer n, $\gcd(n, 12) = 1 \implies \gcd(n, 36) = 1$, we have that $\varphi(36, 12)$ is surjective. Hence, $\operatorname{Im} \varphi(36, 12) = U_{12}$. Moreover, $\operatorname{Ker} \varphi(36, 12)$ comprises all elements in U_{36} that are congruent to 1 modulo 12, and these are precisely, $\{[1], [13], [25]\}$. Note that $\operatorname{Ker} \varphi(36, 12) \cong \mathbb{Z}_3$.

7. Show that $\operatorname{Aut}(D_8) \cong D_8$. [Hint: First, use the fact that an automorphism preserves order of an element to show that $|\operatorname{Aut}(D_8)| \leq 8$. Then realizing that $D_8 \leq D_{16}$, consider the action $D_{16} \curvearrowright^c D_8$.]

   Solution. From class, we know $D_8 = \langle r, s \rangle$, where $o(r) = 4$ and $o(s) = 2$. Hence, any $\varphi \in \operatorname{Aut}(D_8)$ is completely determined by $\varphi(r)$ and $\varphi(s)$. Furthermore, as isomorphisms preserve order, we have that $o(\varphi(r)) = 4$ and $o(\varphi(s)) = 2$. So $\varphi(r) \in \{r, r^3\}$ and $\varphi(s) \in \{s, sr, sr^2, sr^3\}$, which shows that $|\operatorname{Aut}(D_8)| \leq 8$.

   The subgroup $\langle r^2, s \rangle$ of D_{16} is isomorphic to D_8 (why?), and furthermore as $[D_{16} : D_8] = 2$, we have $D_8 \trianglelefteq D_{16}$. Consider the action $D_{16} \curvearrowright^c D_8$, and the induced permutation representation

$$\psi_c : D_{16} \to \operatorname{Aut}(D_8)$$

   with $\operatorname{Ker} \psi_c = C_{D_{16}}(D_8)$. Note that $C_{D_{16}}(D_8) = \{1, r^4\}$, as these are the only elements of D_{16} that commute with both r^2 and s. Thus the action

$$\operatorname{Aut}(D_8) \cong D_{16}/\{1, r^4\} \cong D_8,$$

   as there is a surjective homomorphism

$$\gamma : D_{16} \to D_8$$

   with $\operatorname{Ker} \gamma = \{1, r^4\}$. (What is γ?)